# Uncertainty quantification of diffuse sound insulation values

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**Abstract**
The sound insulation of a partitioning structure depends on the nature of the sound fields in the source and receiver rooms. In order to uniquely define the sound insulation, both sound fields are often taken to be diffuse. A diffuse field model may represent the sound field of a conceptual ensemble of rooms with the same volume and reverberation time, but otherwise any possible arrangement of boundaries and small objects that have a wave scattering effect. Uncertainty due to random wave scattering is therefore inherently present in the computed diffuse transmission loss. In this paper, closed-form expressions are derived for quantifying this uncertainty, such that it becomes possible to estimate by how much the transmission loss of a particular room-wall-room system may deviate from the nominal, ensemble average value which corresponds with omni-directional incidence and halfspace radiation. First, an exact expression is derived, based on the hybrid deterministic-statistical energy analysis theory. Its evaluation requires knowledge of the dynamic stiffness of the partitioning structure and of the radiation impedances of the surrounding fluids. Subsequently, an approximate expression is derived, which require only energetic quantities, making it also applicable in an experimental setting. The accuracies of both expressions are investigated in a numerical study.

**Keywords**: sound insulation, uncertainty quantification, diffuse sound field, frequency averaging.

## 1 Introduction

The airborne sound insulation of a partitioning structure does not only depend on its own dimensions and properties, but also on the nature of the incident sound field at the source side and on that of the radiated sound field at the receiver side. In an indoor setting, both sound fields are conventionally approximated as (fully) diffuse, in the sense that the incident sound field consists of incoherent plane waves coming from all possible directions and carrying the same energy, and the radiated sound field equals that of an acoustic halfspace. This is a good approximation at high frequencies, i.e., above the Schroeder frequency of both rooms, where the modal overlap is very high. At lower frequencies, however, large differences between the sound transmission loss for a particular room-wall-room system and the nominal diffuse transmission loss can occur [1]. Consequently, the prediction of the low-frequency sound insulation of a particular room-wall-room configuration requires detailed models of the acoustic volumes, coupled through a wall model. Such a detailed room-wall-room analysis comes at a substantial modeling and computation cost. Furthermore, very often the performance of the wall across a range of source and receiver positions and across a range of source and receiver rooms is of interest, such that the detailed analysis needs to be repeated multiple times, further increasing the computation cost.

However, if the sound insulation of a given wall across an ensemble of room-wall-room systems is of interest and there is sufficient randomness across that ensemble, then the fact that the ensemble average equals the diffuse sound transmission loss, also at low frequencies, can be exploited [2]. A caveat is of course that, below the Schroeder frequency, the variance across the ensemble can be large, so it should be quantified if one wants to estimate by how much the transmission loss of a particular room-wall-room system may deviate from the nominal, mean diffuse value. Such an estimation is also useful when a particular room-wall-room system is of interest, as it allows the analyst to precisely quantify from which frequency onwards the mean diffuse value is a sufficiently accurate approximation of the transmission loss, rather than relying on the Schroeder frequency.

For the variance quantification, it is important to notice that a diffuse field is by definition a random field, composed of a large number of statistically independent plane waves, the spatial phase of which is uniformly distributed and independent from the amplitude [3]. A diffuse sound field therefore represents not just a nominal situation, but the sound field of a conceptual random ensemble of rooms with the same modal density and total absorption, but otherwise any possible arrangement of boundaries and small objects that have a wave scattering effect. This ensemble has maximum uncertainty, or maximum entropy, across all possible ensembles of rooms with the same modal density and reverberation time [4]. When the sound field in an isolated room can be considered as diffuse, it follows immediately from the above definition that its mode shapes are independent, zero-mean Gaussian random fields. The statistical properties of the natural frequencies relate to those of the eigenvalue spacings Gaussian Orthogonal Ensemble matrix from random matrix theory [5]. Based on this fact, an expression for the variance of the total energy of a diffuse field has been derived [6, 7]. It does not only depend on the modal overlap of the room and on the frequency integration bandwidth, but also on the nature of the loading: for point loading, the variance of the total energy decreases with the number of incoherent sources and it is also lower when the loading is deterministic rather than random. It has found application in the analysis of structure-borne sound radiation into a diffuse field, where closed-form expressions for the variance of the radiated sound power and sound pressure levels have been derived [8]. It can also be employed in the statistical energy analysis of a built-up system if the parameters describing the loading of one particular diffuse subsystem by all other diffuse subsystems can be evaluated [9].

The fact that the diffuse field variance depends rather strongly on the type of excitation, complicates the variance analysis of the sound transmission loss in between two diffuse rooms. At a given frequency, the receiver room may be excited by one, several, or many modes of the vibrating wall. Furthermore, each wall mode is randomly excited by the diffuse sound field at the source side. It is not immediately clear whether and how the excitation parameters of the variance estimation procedure that was developed in [7, 8, 9] might be evaluated for the more complex subsystem excitations considered here. Progress has been made in a hybrid deterministic-statistical energy analysis setting, where diffuse subsystems of finite size can be connected to each other and to deterministic system parts in an arbitrary complex way [10]. In this setting, closed-form expressions have been derived for the variance of the total energies of diffuse subsystems [11, 12]. These fundamental results have been employed successfully for analyzing the diffuse sound transmission loss variance [2, 13, 14]. The related expressions are elaborate and their evaluation requires the halfspace sound radiation impedances of the relevant room modes. Furthermore, they are applicable to the harmonic transmission loss only, while in technical acoustics, transmission losses in frequency bands are usually of interest [1].

In the present work, two developments for diffuse sound transmission loss variance quantification are presented and validated against numerical simulations. The first development concerns the analysis of the band-averaged, rather than the harmonic, transmission loss variance. First, it is demonstrated that the relationship between the diffuse transmission coefficient and the two-room coupling loss factor which is known to hold in the ensemble average sense [15], also holds for every individual member of the random ensemble. Then, the recent hybrid deterministic-SEA band-averaged variance results of [12] are exploited for establishing a new diffuse sound transmission loss variance expression that is applicable to any wall type and to both light and heavy fluid loading. The second development concerns the derivation of a relatively simple formula for the diffuse sound transmission loss that is not only applicable for numerical predictions, but also in an experimental setting. The formula does not require the halfspace radiation impedance of the wall, but involves only quantities that are measured in a conventional sound transmission test. It is derived by introducing approximations into the general diffuse variance expression, which restrict its application to light fluid loading.

## 2 Transmission loss and coupling loss factor

The transmission loss $R$ across a partition wall is defined as

$$R(\omega) := 10 \log \frac{1}{\tau_{12}(\omega)} \quad \text{with} \quad \tau_{12}(\omega) := \frac{P_{\text{in}}^{(1\to 2)}(\omega)}{P_{\text{inc}}^{(1)}(\omega)}, \tag{1}$$

where $\omega$ represents frequency, $\tau_{12}$ the transmission coefficient, $P_{\text{inc}}^{(1)}$ the sound power that is incident upon the wall in the source room (room 1) and $P_{\text{in}}^{(1\to2)}$ the power flow from source to receiver room (room 2). A related quantity is the coupling loss factor $\eta_{12}$ between the source and receiver rooms:

$$\eta_{12}(\omega) := \frac{P_{\text{in}}^{(1\to2)}(\omega)}{\omega E_1(\omega)}, \tag{2}$$

where $E_1$ is the total sound energy in the source room. From these definitions, it follows that

$$\eta_{12}(\omega)\omega E_1(\omega) = \tau_{12}(\omega) P_{\text{inc}}^{(1)}(\omega). \tag{3}$$

The wall is taken to be a linear mechanical system, so $\tau_{12}$ and $\eta_{12}$ do not depend on $P_{\text{inc}^{(1)}}$ nor on $E_1$. For a given incident sound field, the ratio between $E_1$ and $P_{\text{inc}}^{(1)}$ is therefore equal to a constant $C$:

$$E_1(\omega) = C P_{\text{inc}}^{(1)}(\omega) \qquad \text{such that} \qquad \tau_{12}(\omega) = C\omega\eta_{12}(\omega). \tag{4}$$

This last expression is valid for any room-wall-room system. In what follows, it is assumed that the source room carries a diffuse sound field. In that case, the ensemble mean of the incident power relates to the total energy in the room via [15]:

$$\hat{P}_{\text{inc,rev}}^{(1)}(\omega) = \frac{cS}{4V_1}\hat{E}_1(\omega). \tag{5}$$

where the hat denotes the mean across the diffuse random field ensemble, $c$ the sound speed, $S$ the surface area of the wall, and $V_1$ the source room volume. Taking the expected value of both sides of Equation 3 and accounting for the fact that not only $\tau_{12}$ and $P_{\text{inc}}^{(1)}$, but also $\eta_{12}$ and $E_1$ are independent from each other yields

$$\hat{\tau}_{12}(\omega) = C\omega\hat{\eta}_{12}(\omega) \quad \text{with} \quad C = \frac{4V_1}{cS}. \tag{6}$$

From this expression, the constant $C$ which appears in Equation 4 can be determined, such that

$$\tau_{12}(\omega) = \frac{4\omega V_1}{cS}\eta_{12}(\omega). \tag{7}$$

This final expression is known to hold in an ensemble averaged sense [15, Eq. 3.3.22], however, the above derivation demonstrates that it also holds for each individual member of the random ensemble of room-wall-room systems that is being considered here (deterministic wall and random diffuse sound fields in the rooms).

# 3 Prediction of the mean diffuse transmission loss of a complex finite wall

The prediction of the nominal, ensemble averaged transmission loss of a wall that is situated in between two diffuse sound fields is treated in this section. It does not contain new theoretical developments, but rather introduces the concepts, definitions and notations that are necessary for the variance analysis that follows, and this in a more straightforward way than in previous publications such as [2]. The wall is taken to be finite and deterministic and its displacement field is described with a finite set of (generalized) displacement degrees of freedom, which are collected in a frequency-dependent complex vector $\mathbf{q}(\omega)$. Phasor notation is being employed, such that the time history of the wall displacement at frequency $\omega$ is obtained as

$$\mathbf{q}(t) = \text{Re}\left\{\mathbf{q}(\omega)e^{\text{i}\omega t}\right\}. \tag{8}$$

The equation of motion of the wall can be expressed as

$$\mathbf{D}_\text{d}\mathbf{q} = \mathbf{f} + \mathbf{f}_1 + \mathbf{f}_2, \tag{9}$$

where $\mathbf{D}_\mathrm{d}$ represents the deterministic, in-vacuo dynamic stiffness matrix of the wall, $\mathbf{f}$ the external loading on the wall, and $\mathbf{f}_j$ the reaction forces onto the wall caused by the sound pressure field in room $j$ that results from the wall displacement $\mathbf{q}$. The equation of motion of room $j$ reads

$$\mathbf{D}_j\mathbf{q} = -\mathbf{f}_j, \tag{10}$$

where $\mathbf{D}_j$ is the dynamic stiffness matrix of room $j$.

If random wave scattering is present in the rooms, the sound field in each room can be decomposed into a direct sound field, representing waves propagating away from the wall, and a diffuse reverberant sound field, representing scattered waves traveling back to the wall. The equation of motion of the room then reads

$$\left(\mathbf{D}_\mathrm{dir}^{(j)} + \mathbf{D}_\mathrm{rev}^{(j)}\right)\mathbf{q} = -\mathbf{f}_j. \tag{11}$$

The direct field is deterministic, while the diffuse field is random. Since the direct field dynamic stiffness matrix $\mathbf{D}_\mathrm{dir}^{(j)}$ represents that of an acoustic halfspace (or anechoic space), it can be evaluated with the Rayleigh integral and in more general cases with the boundary element method. Substitution into Equation 9 results in

$$\left(\mathbf{D}_\mathrm{tot} + \mathbf{D}_\mathrm{rev}^{(1)} + \mathbf{D}_\mathrm{rev}^{(2)}\right)\mathbf{q} = \mathbf{f}, \qquad \text{where} \qquad \mathbf{D}_\mathrm{tot} := \mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{dir}^{(1)} + \mathbf{D}_\mathrm{dir}^{(2)}. \tag{12}$$

Since the ensemble average of the reverberant dynamic stiffness matrices is zero [16], one may approximate the wall displacement by a first-order perturbation expansion around $\mathbf{D}_\mathrm{rev}^{(1)} = \mathbf{D}_\mathrm{rev}^{(2)} = \mathbf{0}$:

$$\mathbf{q} \approx \mathbf{D}_\mathrm{tot}^{-1}\mathbf{f} - \mathbf{D}_\mathrm{tot}^{-1}\left(\mathbf{D}_\mathrm{rev}^{(1)} + \mathbf{D}_\mathrm{rev}^{(2)}\right)\mathbf{D}_\mathrm{tot}^{-1}\mathbf{f}. \tag{13}$$

The wall displacement results in sound power injection into the receiver room, that can be evaluated as

$$P_\mathrm{in}^{(2)} = \frac{1}{2}\mathrm{Re}\left\{(-\mathrm{i}\omega)\mathbf{q}^\mathrm{H}(-\mathbf{f}_2)\right\} = \frac{\omega}{2}\mathbf{q}^\mathrm{H}\left(\tilde{\mathbf{D}}_\mathrm{dir}^{(2)} + \tilde{\mathbf{D}}_\mathrm{rev}^{(2)}\right)\mathbf{q}, \tag{14}$$

where the imaginary part of a matrix is denoted with a tilde. Substitution of Equation 13 leads to

$$P_\mathrm{in}^{(2)} \approx \frac{\omega}{2}\mathbf{f}^\mathrm{H}\mathbf{D}_\mathrm{tot}^{-\mathrm{H}}\left(\tilde{\mathbf{D}}_\mathrm{dir}^{(2)} + \tilde{\mathbf{D}}_\mathrm{rev}^{(2)}\right)\mathbf{D}_\mathrm{tot}^{-1}\mathbf{f} - \omega\mathrm{Re}\left\{\mathbf{f}^\mathrm{H}\mathbf{D}_\mathrm{tot}^{-\mathrm{H}}\tilde{\mathbf{D}}_\mathrm{dir}^{(2)}\mathbf{D}_\mathrm{tot}^{-1}\left(\mathbf{D}_\mathrm{rev}^{(1)} + \mathbf{D}_\mathrm{rev}^{(2)}\right)\mathbf{D}_\mathrm{rev}^{(2)}\mathbf{D}_\mathrm{tot}^{-1}\mathbf{f}\right\}. \tag{15}$$

If the direct excitation of the wall $\mathbf{f}$ stems from an externally excited diffuse field in room 1 with fixed total energy $E_1$, then $P_\mathrm{in}^{(2)}$ corresponds with $P_\mathrm{in}^{(1\to2)}$ which appears in Eqs. (1-2). The wall is then excited by three diffuse sound pressure fields: a field in room 1 which results from external excitation (represented by the force $\mathbf{f}$), a field in room 1 which results from radiation by the wall and subsequent reflection of the radiated waves in that room (represented by $\mathbf{D}_\mathrm{rev}^{(1)}$), and a similar field in room 2 (represented by $\mathbf{D}_\mathrm{rev}^{(2)}$). Since these diffuse sound fields result from different reflections, they are uncorrelated [11].

With this insight, and acknowledging that the diffuse reverberant system matrices have zero mean, the expected value of Equation 15 can be evaluated to yield the ensemble averaged value of the power flow:

$$\hat{P}_\mathrm{in}^{(1\to2)} = \frac{\omega}{2}\mathrm{Tr}\left(\hat{\mathbf{S}}_\mathrm{ff}\mathbf{D}_\mathrm{tot}^{-\mathrm{H}}\tilde{\mathbf{D}}_\mathrm{dir}^{(2)}\mathbf{D}_\mathrm{tot}^{-1}\right), \qquad \text{with} \qquad \mathbf{S}_\mathrm{ff} := \mathbf{f}\mathbf{f}^\mathrm{H}, \tag{16}$$

and where Tr denotes the matrix trace. The mean cross-spectrum of the diffuse excitation forces $\hat{\mathbf{S}}_\mathrm{ff}$ in room 1 can be evaluated by invoking the so-called diffuse field reciprocity relationship [17]:

$$\hat{\mathbf{S}}_\mathrm{ff} = \frac{4\hat{E}_1}{\omega\pi n_1}\tilde{\mathbf{D}}_\mathrm{dir}^{(1)}, \tag{17}$$

where $n_1$ denotes the modal density in the source room. Substitution of this result into the previous expression and subsequent combination with Eqs. (3,6) results in

$$\hat{\tau}_{12}(\omega) = \frac{8V_1}{cS\pi n_1}\mathrm{Tr}\left(\tilde{\mathbf{D}}_\mathrm{dir}^{(1)}\mathbf{D}_\mathrm{tot}^{-\mathrm{H}}\tilde{\mathbf{D}}_\mathrm{dir}^{(2)}\mathbf{D}_\mathrm{tot}^{-1}\right) \tag{18}$$

The nominal, ensemble average diffuse field transmission coefficient can be obtained from this last expression, hence also the transmission loss. Its evaluation requires the dynamic stiffness matrix of the wall and that of the direct field of both rooms next to readily available quantities such as the room volume and modal density.

The above analysis is in principle valid only for harmonic transmission, i.e., where $\tau_{12}$ represents the transmission loss at a single frequency $\omega$ rather than over a frequency band $\Delta = \omega_{\mathrm{u}} - \omega_{\mathrm{l}}$. In the latter, case, the transmission loss and coefficient are defined as

$$R_\Delta(\omega) := 10 \log \frac{1}{\tau_{12,\Delta}(\omega)} \quad \text{with} \quad \tau_{12,\Delta}(\omega) = \frac{1}{\Delta} \int_{\omega_{\mathrm{l}}}^{\omega_{\mathrm{u}}} \tau_{12}(\omega) \mathrm{d}\omega = \frac{\int_{\omega_{\mathrm{l}}}^{\omega_{\mathrm{u}}} P_{\mathrm{in}}^{(1\to2)}(\omega)\mathrm{d}\omega}{P_{\mathrm{inc},\Delta}^{(1)}(\omega)\Delta} \tag{19}$$

where in the last equality, the fact that the prescribed incident power $P_{\mathrm{inc},\Delta}^{(1)}$ is by definition constant over the considered frequency band, has been employed. Taking the expected value of these expressions and substituting Equation 18 leads directly to the ensemble average quantities $\hat{\tau}_{12,\Delta}$ and $\hat{R}_\Delta$.

The approach that was presented here for evaluating the diffuse sound transmission loss, is generally applicable and does not require any assumption on the geometry and dynamic behavior of the wall other than linearity. In contrast, in the conventional approach to diffuse sound transmission prediction, the wall is considered to be of infinite extent, the transmission coefficient is computed for a plane incident sound wave and halfspace reflection (at the source side) and radiation (at the receiver side), and subsequently integrated over all angles of incidence to yield the mean diffuse transmission loss $\tau_{12}$ [1, 18]. The maximum angle of attack is usually limited to a cut-off value that is smaller than 90° on an empirical basis, in order to have better correspondence with measured values [18]. Problems of this kind are avoided in the approach that was presented here, because the finite extent of the interface between wall and rooms is accurately accounted for.

# 4 Prediction of the variance of the diffuse transmission loss

## 4.1. General case

The variance of $R_{12,\Delta}$ follows from a first-order approximation of its definition:

$$\mathrm{Var}\left[R_\Delta\right] \approx \left(\frac{\mathrm{d}R_\Delta}{\mathrm{d}\tau_{12,\Delta}}\right)^2 \mathrm{Var}\left[\hat{\tau}_{12,\Delta}\right] = \left(\frac{10}{\ln 10}\right)^2 \frac{\mathrm{Var}\left[\tau_{12,\Delta}\right]}{\hat{\tau}_{12,\Delta}^2} = \left(\frac{10}{\ln 10}\right)^2 \frac{\mathrm{Var}\left[P_{\mathrm{in},\Delta}^{(1\to2)}\right]}{\hat{P}_{\mathrm{in},\Delta}^{(1\to2)}}. \tag{20}$$

The variance of the band-integrated transmitted power $P_{\mathrm{in},\Delta}^{(1\to2)}$ results from the integration of the cross-covariance between the harmonic transmitted powers at different frequencies within the considered band:

$$\mathrm{Var}\left[P_{\mathrm{in},\Delta}^{(1\to2)}\right] = \int_{\omega_{\mathrm{l}}}^{\omega_{\mathrm{u}}} \int_{\omega_{\mathrm{l}}}^{\omega_{\mathrm{u}}} \mathrm{Cov}\left[P_{\mathrm{in}}^{(1\to2)}(\omega_1) P_{\mathrm{in}}^{(1\to2)}(\omega_2)\right]. \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \tag{21}$$

Substitution of Equation 15 results in a second-order approximation of the variance. It can be further evaluated by noting that the reverberant force vectors that relate to the external excitation and the responses of the source and receiver rooms ($\mathbf{f}$, $\mathbf{D}_{\mathrm{rev}}^{(1)}$ and $\mathbf{D}_{\mathrm{rev}}^{(2)}$, respectively) are mutually uncorrelated and circular complex Gaussian distributed, and by assuming that their statistical properties as well as the properties of the wall are approximately constant within the frequency band $\Delta$. This evaluation then follows the same lines as that of the variance of the entries of the power balance matrix in a hybrid deterministic-statistical energy analysis in frequency bands, which was presented in detail by Reynders and Langley [12, Secs. 4.3 and 5.2]. As a result, the relative variance of $P_{\mathrm{in},\Delta}^{(1\to2)}$ is obtained by evaluation of [12, Eq. (62)]. Some further algebra leads to

$$\begin{aligned} \frac{\mathrm{Var}\left[\tau_{12,\Delta}\right]}{\hat{\tau}_{12,\Delta}^2} 4\hat{\eta}_{21}^2 = & \sum_{j=1}^{2} \frac{b_j}{\pi m_j} \mathrm{Re}\left(\mathrm{Tr}\left(\mathbf{H}_{1j}\mathbf{G}^{(2)}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(j)}\mathbf{G}^{(2)}\right)\right) + \frac{2\hat{\eta}_{21} b_{1\mathrm{R}2}}{\pi m_1} \mathrm{Re}\left(\mathrm{Tr}\left(\mathbf{H}_{11}\mathbf{G}^{(2)}\right)\right) \\ & + \frac{b_{2\mathrm{R}2}}{\pi m_2} \mathrm{Re}\left(\mathrm{Tr}\left(\mathbf{H}_{12}\mathbf{G}^{(2)}\right) \mathrm{Tr}\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(2)}\mathbf{G}^{(2)}\right)\right) + b_2 \mathrm{Re}\left(\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^2\right)\right) \end{aligned} \tag{22}$$

where $m_j := \omega\eta_j n_j$ represents the modal overlap of room $j$, and

$$b_j := \frac{-1}{B_j^2}\ln\left(1+B_j^2\right)+\frac{2}{B_j}\mathrm{atan}\left(B_j\right) \qquad \text{with} \qquad B_j := \frac{\Delta}{\omega\eta_j} \tag{23}$$

$$b_{j\mathrm{R2}} := \delta_{2j}\frac{\mathrm{atan}\left(B_j\right)}{B_j}+(1-\delta_{2j})\frac{B_2^2 b_2 - B_j^2 b_j}{B_2^2 - B_j^2} \tag{24}$$

$$\mathbf{G}^{(j)} := \frac{4}{\omega\pi n_j}\mathbf{D}_{\mathrm{tot}}^{-1}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(j)}\mathbf{D}_{\mathrm{tot}}^{-\mathrm{H}} \tag{25}$$

$$\mathbf{H}_{1j} := \omega\pi n_j\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(j)}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}+\delta_{1j}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}-2\delta_{1j}\mathrm{i}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{D}_{\mathrm{tot}}^{-1}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}+2\delta_{1j}\mathrm{i}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{D}_{\mathrm{tot}}^{-H}\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)} \tag{26}$$

In these expressions, $\delta_{2j}$ denotes the Kronecker delta, i.e., $\delta_{2j} = 0$ when $j = 1$ and $\delta_{2j} = 1$ when $j = 2$. The harmonic variance result is recovered by reducing the bandwidth $\Delta \to 0$, in which case $b_j \to 1$ and $b_{s\mathrm{R2}} \to 1$.

### 4.2. Light fluid loading

A substantial simplification can be achieved for the common special case of light fluid loading, i.e., when the wall impedance is much larger than the radiation impedance of the acoustic fluids in both rooms. The second term of $\mathbf{H}_{11}$ is then the dominant one:

$$\mathbf{H}_{11} \approx \tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}. \tag{27}$$

Furthermore, in most practical situations, the direct field dynamic radiation impedances of both rooms are similar, such that

$$\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)} \approx \tilde{\mathbf{D}}_{\mathrm{dir}}^{(2)} \tag{28}$$

With these approximations, and the application of the definition of $\mathbf{H}_{12}$, Equation 22 transforms into

$$\frac{\mathrm{Var}\left[\tau_{12,\Delta}\right]}{\hat{\tau}_{12,\Delta}^2} \approx \frac{b_2\omega n_2}{m_2 4\hat{\eta}_{21}^2}\mathrm{Re}\left(\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^3\right)\right)+\frac{b_{1\mathrm{R2}}}{\pi m_1}+\frac{b_{2\mathrm{R2}}\omega n_2}{m_2 2\hat{\eta}_{21}}\mathrm{Re}\left(\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^2\right)\right)$$
$$+\frac{1}{4\hat{\eta}_{21}^2}\left(\frac{b_1}{\pi m_1}+b_2\right)\mathrm{Re}\left(\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^2\right)\right). \tag{29}$$

Without loss of generality, one may assume that the wall displacement is expressed in terms of its in-vacuo modal coordinates, i.e., the elements of $\mathbf{q}$ represent the modal displacements of the wall. The dynamic stiffness matrix $\mathbf{D}_{\mathrm{d}}$ of the wall is then a diagonal matrix when the wall damping is proportional. Furthermore, if the fluid loading is light, it can be assumed that the acoustic fluid does not couple the in-vacuo modes of the structure, so the matrices $\mathbf{D}_{\mathrm{dir}}^{(j)}$ will be diagonal, and therefore also $\mathbf{D}_{\mathrm{tot}}$ and $\mathbf{G}^{(j)}$. Finally, if there are $N$ modes that contribute to the sound transmission in the considered frequency band, and those modal contributions are approximately equal, then the diagonal elements of $\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}$ are approximately equal to each other, so

$$\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^k\right) \approx \frac{1}{N^{k-1}}\left(\mathrm{Tr}\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)\right)^k = \frac{2^k}{N^{k-1}}\hat{\eta}_{21}^k. \tag{30}$$

Subsitution of this result into Equation 29 and rearranging terms leads to

$$\frac{\mathrm{Var}\left[\tau_{12,\Delta}\right]}{\hat{\tau}_{12,\Delta}^2} \approx \frac{b_2}{N}+\frac{1}{\pi m_1}\left(b_{1\mathrm{R2}}+\frac{b_1}{N}\right)+\frac{2\hat{\eta}_{21}}{N\eta_2}\left(b_{2\mathrm{R2}}+\frac{b_2}{N}\right). \tag{31}$$

Because of its relatively simple form, this final expression lends itself easily to physical interpretation. In the generic case where the energy loss in the receiver room is dominated by absorption rather than transmission through the wall, $\eta_{21} \ll \eta_2$, and the third term of the variance expression is negligibly small with respect to the first two terms. The second term is inversely proportional to the modal overlap factor of the source room, $m_1$, so it is expected to be the dominant term at low frequencies. However, if $\eta_1$ is independent of frequency, $m_1$ increases with the third power of frequency, such that at higher frequencies, the first term will normally be the dominant one.

The variables $b_1$, $b_2$, $b_{1R2}$ and $b_{2R2}$ which appear in this expression, depend only on ratios of the relative frequency bandwidth and the loss factors of the rooms, $B_1$ and $B_2$. For very small bandwidths, the variables all approximate unity. They decrease monotonically with increasing relative frequency bandwidth, and with decreasing loss factor or, equivalently, with increasing reverberation time.

The observations related to Equation 31 are in good agreement with known results. For example, the fact that the modal overlap of the source room is the dominant source of uncertainty at low frequencies, is reflected in the practice to use the largest of both rooms in a transmission suite as source room, as prescribed by the ISO 10140-5 standard. Furthermore, it is well known that the inter-laboratory variability of diffuse transmission loss values is particularly large for walls with a low modal density, i.e., for low values of $N$ [19].

Equation 31 is much easier to evaluate than its more general counterpart, Equation 22, as it only depends on the following quantities: the loss factors of the rooms (or equivalently, their reverberation times), the modal overlap of the source room, the frequency bandwidth, the coupling loss (or, equivalently, the transmission loss) itself, and the number of wall modes contributing the sound transmission. These quantities can all be easily obtained from experiments, or alternatively from simple analytical predictions, such that Equation 31 can be applied in a much wider setting than the sound insulation prediction framework in which it was derived.

The only exception is perhaps the number of modes $N$ that correspond to the sound transmission. If individual modal resonances of the wall dominate the sound transmission, then $N = 1$. This is not only the case at wall resonance frequency dips of the harmonic transmission loss, but also for the corresponding band-averaged values, on which such dips have a decisive influence. In between resonance dips, several wall modes are excited, so $N \geq 2$. In the next section, a simple expression that can be used for estimating $N$, is derived. It depends only on the modal overlap of the wall.

### 4.3. Approximate estimation of the number of modes $N$

This section elaborates on the computation of the variable $N$ that was introduced in Equation 30. The main idea is to elaborate the first equality in Equation 30 for the special case $k = 2$. With the assumptions made so far, this leads in the first instance to

$$N = \frac{\left(\mathrm{Tr}\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)\right)^2}{\mathrm{Tr}\left(\left(\tilde{\mathbf{D}}_{\mathrm{dir}}^{(1)}\mathbf{G}^{(2)}\right)^2\right)} \approx \frac{\left(\sum_j \frac{\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^2}{\left|D_{\mathrm{d},jj}\right|^2}\right)^2}{\sum_j \frac{\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^4}{\left|D_{\mathrm{d},jj}\right|^4}}. \tag{32}$$

This expression depends both on the natural frequencies of the wall and the modal radiation impedances of the rooms. In situations where these quantities are not (precisely) known, such as standardized measurement situations, progress can be made by approximating the dynamic stiffness of the wall to the ensemble average dynamic stiffness of a wall carrying a diffuse vibration field. Assuming that $\tilde{D}_{\mathrm{dir},jj}^{(1)}$ can be considered as constant in the neighbourhood of the considered frequency, results from random point process theory [6] can be used to obtain the ensemble averages of the numerator and denominator in Equation 32:

$$\mathrm{E}\left[\sum_j \frac{\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^4}{\left|D_{\mathrm{d},jj}\right|^4}\right] \approx \frac{\pi n_{\mathrm{d}}}{4\eta_{\mathrm{d}}^3\omega^7}\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^4, \tag{33}$$

$$\mathrm{E}\left[\left(\sum_j \frac{\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^2}{\left|D_{\mathrm{d},jj}\right|^2}\right)^2\right] \approx \frac{\pi^2 n_{\mathrm{d}}^2}{4\eta_{\mathrm{d}}^2\omega^6}\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^4\left\{1 + \mathrm{relVar}\left(\sum_j \frac{\left(\tilde{D}_{\mathrm{dir},jj}^{(1)}\right)^2}{\left|D_{\mathrm{d},jj}\right|^2}\right)\right\}. \tag{34}$$

Substitution into Equation 32 yields

$$N = \pi m_\mathrm{d} \left\{ 1 + \mathrm{relVar}\left( \sum_j \frac{\left(\tilde{D}^{(1)}_{\mathrm{dir},jj}\right)^2}{\left|D_{\mathrm{d},jj}\right|^2} \right) \right\}, \tag{35}$$

with $m_\mathrm{d} = \omega\eta n_\mathrm{d}$ the modal overlap factor and $n_\mathrm{d}$ the modal density of the wall. The relative variance in this expression has previously been investigated in [6]. As the natural frequency spacings converge to that of the Gaussian Orthogonal Ensemble (GOE), the relative variance equals:

$$\mathrm{relVar}\left( \sum_j \frac{\left(\tilde{D}^{(1)}_{\mathrm{dir},jj}\right)^2}{\left|D_{\mathrm{d},jj}\right|^2} \right) = \frac{1}{\pi m_\mathrm{d}} \left\{1 + q(m_\mathrm{d})\right\} \tag{36}$$

with

$$q(m_\mathrm{d}) = -1 + \frac{1}{2\pi m_\mathrm{d}} \left[1 + \exp(-2\pi m_\mathrm{d})\right] + E_1(\pi m_\mathrm{d}) \left[\cosh(\pi m_\mathrm{d}) - \frac{1}{\pi m_\mathrm{d}} \sinh(\pi m_\mathrm{d})\right]. \tag{37}$$

The function $q(m_\mathrm{d})$ goes to zero at low values and to one at high values of the modal overlap factor $m_\mathrm{d}$. This means that the investigated relative variance equals $1/(\pi m_\mathrm{d})$ at low modal overlap and decreases to zero at high modal overlap, which also implies that $N$ approximately equals $1 + \pi m_\mathrm{d}$ at low and $\pi m_\mathrm{d}$ at high modal overlap. As at high modal overlap, $\pi m_\mathrm{d} \gg 1$, the following approximation can be used for the entire frequency range:

$$N \approx 1 + \pi m_\mathrm{d}. \tag{38}$$

## 5 Numerical validation

The proposed method is validated by comparison with a detailed Monte Carlo model. The sound transmission for a particular wall is obtained by coupling it to two rooms (the source and receiver room). A total of 30 point air pockets (or acoustic point masses) are distributed at random locations within the room. The air pockets are highly idealized models for small wave scatterers in the room, in the same way as point masses would be highly idealized models for small wave scatterers on a plate. Each air pocket has 0.4 % of the total acoustic mass $V/c^2$ of the room. The probability distribution of the location of each air pocket is uniform throughout the entire room, and the locations of the air pockets are statistically independent. From a certain frequency onwards, the room with randomly distributed acoustic masses behaves as a diffuse field. The wall, on the other hand, is modelled deterministically.

The considered wall has a thickness of 10 cm. It consists of gypsum blocks with mass density $\rho = 910\,\mathrm{kg/m^3}$, Young's modulus $E = 3.15\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.2$, and damping loss factor $\eta = 0.03$. The source room has dimensions $V_1 = 3.12\,\mathrm{m} \times 4.32\,\mathrm{m} \times 4.08\,\mathrm{m} \approx 55\,\mathrm{m^3}$ and reverberation time $T_1 = 1.5\,\mathrm{s}$ and the receiver room has dimensions $V_2 = 3.12\,\mathrm{m} \times 4.24\,\mathrm{m} \times 3.78\,\mathrm{m} \approx 50\,\mathrm{m^3}$ and reverberation time $T_2 = 1.5\,\mathrm{s}$.

Fig. 1 displays the mean and variance of the harmonic and 1/3 octave band transmission loss in the frequency range 50 − 500 Hz. It can be noted that both the ensemble average and the variance of the harmonic sound transmission loss as computed from the detailed Monte-Carlo model are well approximated by the diffuse approximations presented in this paper, except at very low frequencies (below 125 Hz). This is because at very low frequencies, where the local harmonic sound fields in the rooms are not very sensitive to the presence of the small random wave scatterers (point air pockets). Also for the mean 1/3-octave band predictions, a good agreement is obtained from 125 Hz onwards. The band-averaged variance however is underestimated when the general expression (22) or the approximate one (31) with Equation 38 are employed. The reason for this is that the harmonic transmission loss varies strongly with frequency, while in the variance theory, the wall properties are taken to be approximately constant within the considered frequency bands. Much better agreement is achieved when setting $N = 1$ in the approximate expression, as the band-averaged transmission loss values are dominated by the individual transmission loss resonance dips within the corresponding frequency bands.

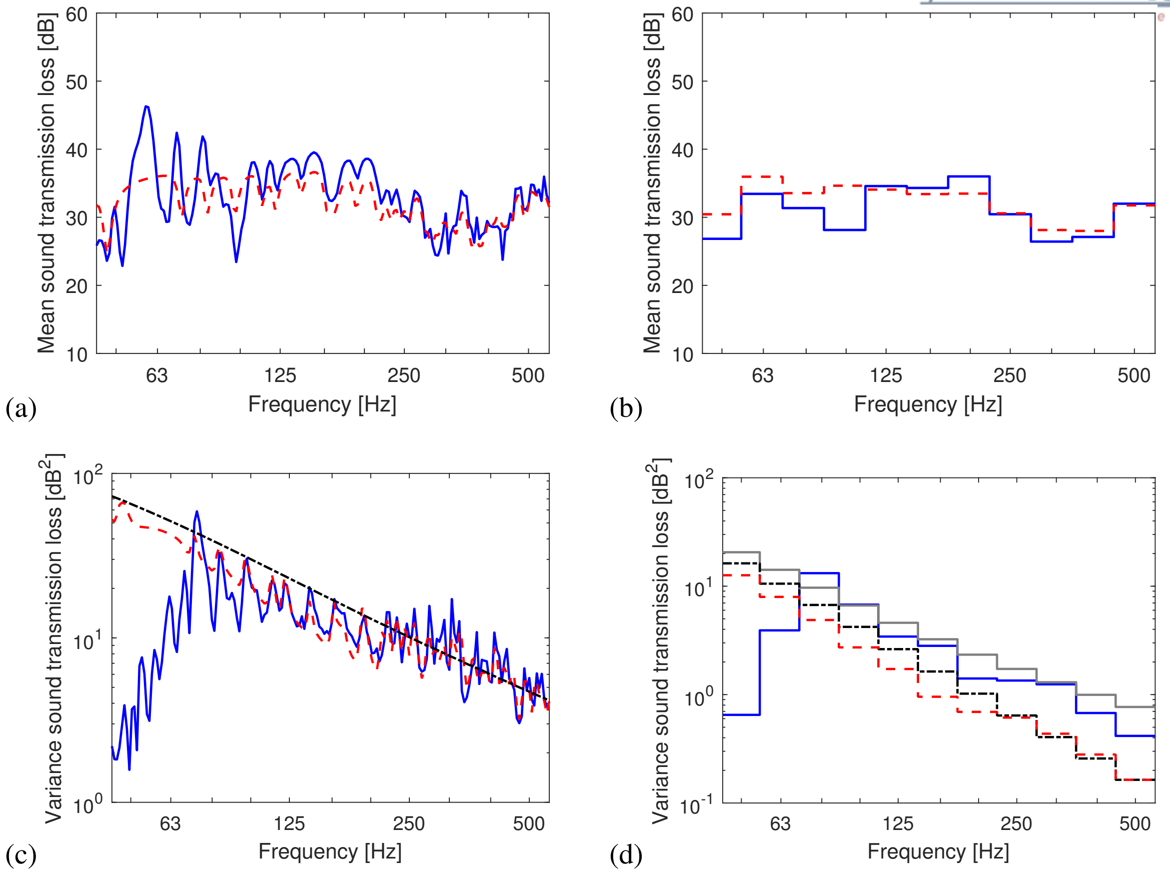


Figure 1: Sound transmission loss of a gypsum block wall: (a,b) mean and (c,d) variance of the (a,c) harmonic and (b,d) 1/3-octave band values. Blue solid lines: detailed Monte Carlo predictions; red dashed lines: general diffuse predictions; black dash-dotted lines: approximate diffuse predictions with $N$ obtained from Eq. 38; grey solid lines: same but with $N = 1$.

# 6 Conclusions

Closed-form expressions have been derived for quantifying the uncertainty of sound insulation values that is inherent in the assumption that the source and receiver rooms carry diffuse fields. A first, generally applicable expression requires knowledge of the dynamic stiffness of the partitioning structure and of the radiation impedances of the surrounding fluids. A second expression that is applicable to cases of light fluid loading requires only energetic quantities, making it also applicable in an experimental setting. A numerical validation confirmed that these expressions can accurately quantify the harmonic sound transmission loss variance of a wall across an ensemble of transmission suites with wave scattering elements placed at random locations. The band-averaged variance can also be relatively well captured when the fact that modal resonance dips dominate the band-averaged transmission loss is accounted for.

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