FAST ACOUSTICAL MODELLING OF ENCLOSURES WITH DIFFUSELY REFLECTING SURFACES

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ABSTRACT

Reliable, fast room acoustics modelling techniques are required both for room design purposes and for entertainment implementations. Techniques like ray-tracing or image source methods, may be rather time consuming.

The complex sound reflection behaviour of enclosure surfaces is usually idealised as being formed by specular reflection components and by lambertian reflection components.

This paper will report on a statistical method that applies to lambertian enclosures, since it is believed that the diffuse reflected components are predominant in real enclosures.

It will be shown that the method presented herein is of fast computation.

INTRODUCTION

This paper presents a statistical method that applies to lambertian enclosures, since it is believed that the diffuse reflected components are predominant in real complex-shaped enclosures. The basis of the method has been already presented by the authors [1].

The objective is to set up a simple calculation technique with low computation time, applicable to a wide range of geometrical configurations. The sound field inside the enclosures is considered to be formed by a large number of sound particles, that are radiated from one, or several, sources. The evolution of the energy densities over the enclosure surfaces is determined by an equation of motion that considers the transition amplitudes, when the sound particles change their location inside the enclosure. Physical phenomena such as sound absorption in the air and absorption at the boundaries are included.

THEORY

In the following work, it will be considered that the sound energy inside an enclosure *S*, of arbitrary shape, is carried by point particles of sound that are emitted from acoustic sources.

*B(**s**, 0)* will be defined as the sound energy density, per unit time and per unit area, at point *R*, located over the surface of the enclosure, and given by the position vector ***s*** at the initial time *t* = 0, with the normalization condition that:

$$\iint_S B(\mathbf{s},\, 0)\, d\mathbf{s} = P \tag{1}$$

where P is the total acoustic power inside the enclosure at *t* = 0. Therefore, *B(**s**, 0) dS* yields the sound energy over an infinitesimal surface element *dS*, with position vector ***s***, at the initial time *t* = 0.

The evolution of this energy density, representing a large number of sound particles, can be determined using the statistical physics' master equation, as described in detail in [1]:

$$\begin{aligned} B(\mathbf{s},\, t) = \iint_S \ldots \iint_S & B(\mathbf{s_1},\, 0)\, W_{dt}^{0}(\mathbf{s_1} \to \mathbf{s_2})(1-a(\mathbf{s_1}))e^{-m\|\mathbf{s_1}-\mathbf{s_2}\|} W_{dt}^{0}(\mathbf{s_2} \to \mathbf{s_3})(1-a(\mathbf{s_2}))e^{-m\|\mathbf{s_2}-\mathbf{s_3}\|}\ldots \\ & W_{dt_n}^{dt_{n-1}}(\mathbf{s_n} \to \mathbf{s})(1-a(\mathbf{s_n}))e^{-m\|\mathbf{s_n}-\mathbf{s}\|} d\mathbf{s_1} d\mathbf{s_2} \ldots d\mathbf{s_n} \end{aligned} \tag{2}$$

where $W_{dt_n}^{dt_{n-1}}(\mathbf{s_1} \to \mathbf{s_2})$ means the amplitude density per unit time that the transition of energy from position ***s₁*** to position ***s₂*** occurs during the time $\sum_{k=1}^{n-1} dt_k$ and $\sum_{k=1}^{n} dt_k$, and where the factor *m* stands for the air absorption coefficient [2]:

$$m = 5.5 \times 10^{-4}\, \frac{50}{h} \left(\frac{f}{1000} \right)^{1.7} \tag{3}$$

for the sound frequency *f* and a relative humidity *h*.

The product of the transition amplitudes in equation (2) means that successive transitions are considered as independent, depending only on the immediate previous transition, which is the case for Markov processes as described by the master equation.

The time-dependent amplitude transition densities $W_{dt_n}^{dt_{n-1}}(\mathbf{s_i} \to \mathbf{s_j})$ can be defined from time-independent amplitude transition densities $T(\mathbf{s_i} \to \mathbf{s_j})$ through formal separation of variables [3]:

$$\begin{aligned} B(\mathbf{s},\, t) = \iint_S \ldots \iint_S & B(\mathbf{s_1},\, 0)\, T(\mathbf{s}_1 \to \mathbf{s}_2)\, d\left(dt \geq \frac{\|\mathbf{s}_1 - \mathbf{s}_2\|}{v} \right) r(\mathbf{s}_1, \mathbf{s}_2) \times \\ & T(\mathbf{s}_2 \to \mathbf{s}_3)\, d\left(dt_2 \geq \frac{\|\mathbf{s}_2 - \mathbf{s}_3\|}{v} \right) r(\mathbf{s}_2, \mathbf{s}_3) \\ & \ldots T(\mathbf{s}_n \to \mathbf{s})\, d\left(dt_n \geq \frac{\|\mathbf{s}_n - \mathbf{s}\|}{v} \right) r(\mathbf{s}_n, \mathbf{s}_n)\, d\mathbf{s_1} d\mathbf{s_2} \ldots d\mathbf{s_n} \end{aligned} \tag{4}$$

where the Boolean function d is defined as:

$$d(True)=1 \qquad (5)$$
$$d(False)=0$$

and $r(\mathbf{s_i},\mathbf{s_j})=(1-a(\mathbf{s_i}))e^{-m\|\mathbf{s}_i-\mathbf{s}_j\|}$

For lambertian enclosures [4]:

$$T(\mathbf{s_i}\rightarrow\mathbf{s_j})=\frac{\cos J_i\,\cos J_j}{p\|\mathbf{s_i}-\mathbf{s_j}\|^2} \qquad (6)$$

where J_i is the angle between $\mathbf{s}_i - \mathbf{s}_j$ and the enclosure's normal at point $\mathbf{s}_i$, and J_j is the angle between $\mathbf{s}_i - \mathbf{s}_j$ and the normal at point $\mathbf{s}_j$. v is the velocity of sound.

<u>Kuttruff's Integral Equation</u>

Kuttruff's Integral Equation can be obtained from the Markovian master equation, as will be shown next.

The non-trivial solution of equation (4) can only be obtained if the delta functions do not force the integrals being all equal zero. Therefore, this condition can be written as:

$$dt=\frac{\|\mathbf{s_1}-\mathbf{s_2}\|}{v};dt_2=\frac{\|\mathbf{s_2}-\mathbf{s_3}\|}{v};\cdots;dt_n=\frac{\|\mathbf{s_n}-\mathbf{s}\|}{v}$$
$$t=\sum_{k=1}^{n}dt_k=\frac{\|\mathbf{s_1}-\mathbf{s_2}\|+\|\mathbf{s_2}-\mathbf{s_3}\|+\cdots+\|\mathbf{s_n}-\mathbf{s}\|}{v}=\frac{R_{TOT}}{v} \qquad (7)$$

where R_{TOT} is the total length travelled by the sound particles. Equation (4) thus becomes:

$$B(\mathbf{s},\,t)=\iint_S\cdots\iint_S B(\mathbf{s_1},\,t-\frac{R_{TOT}}{v})T(\mathbf{s_1}\rightarrow\mathbf{s_2})\,r(\mathbf{s_1},\mathbf{s_2})T(\mathbf{s_2}\rightarrow\mathbf{s_3})\,r(\mathbf{s_2},\mathbf{s_3}) \qquad (8)$$
$$..T(\mathbf{s_n}\rightarrow\mathbf{s})\,r(\mathbf{s_n},\mathbf{s_n})d\mathbf{s_1}d\mathbf{s_2}..d\mathbf{s_n}$$

which is identical to Kuttruff's time-dependent integral equation [5] applied successively to several reflections.

<u>Homogeneous Markov Chain of First Order</u>

If the transition time intervals are assumed to be equal, i.e. $dt = dt_1 = dt_n = t$, where t is a reference time interval, then:

$$t=k\times dt=kt \qquad (9)$$

If, additionally, it is assumed that the distances $\|\mathbf{s_i}-\mathbf{s_j}\|$ can be approximately given by $l=vt$, then equation (4) can be written as:

$$B(\mathbf{s},\,kt)=\underbrace{\iint_S\cdots\iint_S}_{kTimes}B(\mathbf{s_1},\,0)T(\mathbf{s_1}\rightarrow\mathbf{s_2})\,r(\mathbf{s_1},\mathbf{s_2})T(\mathbf{s_2}\rightarrow\mathbf{s_3})\,r(\mathbf{s_2},\mathbf{s_3}) \qquad (10)$$
$$..T(\mathbf{s_k}\rightarrow\mathbf{s})\,r(\mathbf{s_k},\mathbf{s})d\mathbf{s_1}d\mathbf{s_2}..d\mathbf{s_k}$$

that can be rewritten in operator form as follows:

$$B(\mathbf{s}, k\,t) = \Theta^k B(\mathbf{s}, 0) \tag{11}$$

where one can introduce the integral operator Q with kernel K defined through the application:

$$\begin{aligned}
&\Theta : B \mapsto \iint_S B(\mathbf{s_1}, 0)\, T(\mathbf{s_1} \to \mathbf{s})\, r(\mathbf{s_1}, \mathbf{s})\, d\mathbf{s_1} \\
&\Theta : B \mapsto \iint_S B(\mathbf{s_1}, 0)\, K(\mathbf{s_1}, \mathbf{s})\, d\mathbf{s_1} \\
&K(\mathbf{s_1}, \mathbf{s}) = T(\mathbf{s_1} \to \mathbf{s})\, r(\mathbf{s_1}, \mathbf{s})
\end{aligned} \tag{12}$$

Equation (11) together with the assumption (9) represents a homogeneous Markov chain of first order for the time evolution of the energy density *B*.

If we consider the steady-state situation, then it is necessary to sum all the contributions of all sound particles undergoing an infinite number of reflections, thus obtaining a Neumann series:

$$B(\mathbf{s}) = \sum_{k=0}^{\infty} \Theta^k B(\mathbf{s}, 0) \tag{13}$$

Due to the property that the norm of the operator Q is always less than 1 [3], the theorem of Banach of the inverse operator [6] states that there exists an inverse operator, also bounded, and therefore:

$$B(\mathbf{s}) = \sum_{k=0}^{\infty} \Theta^k B(\mathbf{s}, 0) = [I - \Theta]^{-1} B(\mathbf{s}, 0) \tag{14}$$

where *I* is the identity operator.

The equations for the energy density *B* can be discretized by assuming that the entire surface *S* is divided into *M* homogeneous and finite surfaces S_j over which the energy density is constant. In this case, the above multiple surface integrals in equation (10) are converted into *M* sums:

$$\begin{aligned}
B(S_j, kt) = \frac{1}{S_j} \sum_{a1=1}^{M} \dots \sum_{ak=1}^{M} & B(S_{a1}, 0) F(S_{a1} \to S_{a2})\, r(S_{a1}, S_{a2}) F(S_{a2} \to S_{a3})\, r(S_{a2}, S_{a3}) \\
& \dots F(S_{ak} \to S_j)\, r(S_{ak}, S_j)
\end{aligned} \tag{15}$$

where the form factors *F* have been introduced:

$$F(S_i \to S_j) = F_{ij} = \int_{S_i}\int_{S_j} \frac{\cos J_i \cos J_j}{p \left\| \mathbf{s_i} - \mathbf{s_j} \right\|^2} d\mathbf{s_i}\, d\mathbf{s_j} \tag{16}$$

and where $r(S_i, S_j) = r_{ij} = (1 - a(S_i))e^{-mD(S_i, S_j)}$ and *D* refers to the mean distance between surface S_i and surface S_j. In this case, the integral operator Q is converted into:

$$\Xi : B \mapsto \sum_{i=1}^{M} B(S_i, 0) F_{ij}\, r_{ij} = \sum_{i=1}^{M} B(S_i, 0)\, \Xi_{ij} \tag{17}$$

and the equation for the evolution of the energy density can be written in matrix form:

$$[B]_{(kt)} = [\Xi]^k [B]_{(0)} \tag{18}$$

which represents a discrete homogeneous Markov chain of first order [7]. $[B]_{(kt)}$ represents a *M*-dimensional column vector with entries

$$B_{j,k} = B(S_j, kt) \tag{19}$$

which define the energy density over S_j at time $k\tau$. $[B]_{(0)}$ is a M-dimensional column vector, called the starting vector:

$$[B]_{(0)} = \lfloor B_{1,0}, B_{2,0}, B_{3,0}, \ldots, B_{M,0} \rfloor \tag{20}$$

whose entries represent the initial acoustical energy densities (or sound particle density distribution) over the various surfaces of the enclosure at t=0. $[\Xi]^k$ represents the k^{th} matrix power of the $M \times M$ matrix Ξ, whose entries are defined by (17).

The starting vector $[B]_{(0)}$ can be obtained by considering spherical waves radiated from N omni-directional sound sources.

For the reference time, τ, the classical mean transition time can be chosen [8]:

$$\tau = \frac{4V}{vS} \tag{21}$$

where V is the total volume of the enclosure.

The steady-state intensity of the sound at the receiving point $\mathbf{s_r}$ can be calculated admitting lambertian radiation from the walls. The total steady-state mean value, I_r, can then be given by [1]:

$$\begin{aligned} I_r(\mathbf{s_r}) &= \sum_{j=1}^{M}\sum_{k=0}^{\infty} I_j^k(\mathbf{s_r}) \\ &= \sum_{j=1}^{M}\sum_{k=0}^{\infty} \frac{B_{j,k}\Omega_j \times (1-\alpha_j)}{\pi} \end{aligned} \tag{22}$$

The energy decay can be determined by considering the various k transitions over a time interval $k\tau$ [1]:

$$I_r(k\tau) = \sum_{j=1}^{n} \frac{\Pi_{j,k}\Omega_j (1-\alpha_j)}{\pi} \tag{23}$$

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