PHASE SPACE ANALYSIS FOR A HIGH FREQUENCY TIME-HARMONIC ACOUSTIC FIELD IN A PLANAR WAVEGUIDE.

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ABSTRACT
The aim of this paper is to present a new description of a two-dimensional high frequency acoustic field by means of a conjoint «space-wave number » representation. To a signal with spatial coordinate (x,y), is associated a function defined in the phase space domain (x,y,k_x,k_y). At each point (x,y) of the field, is asociated a two-dimensional wave number spectrum. Those representations are given by quadratic phase space distribution and are used to analyse, for a given point, the acoustic wave field created by an harmonic source point located between two infinite rigid walls. The result is the contribution of different wave vectors which contribute to the field value at the analysis point.

I. INTRODUCTION

The present study consists of proposing a new tool for local analysis of time harmonic acoustic wave field in a two-dimensional space (a time dependence $e^{-j\omega t}$ is assumed and suppressed in the following). It is sometimes convenient to describe a space varying signal with complex amplitude $\mathrm{A}(\mathbf{r})$ not in the space domain, but in the wave number domain by means of it wave number spectrum $\mathbf{k}$. For example, the Fourier transform of the function $\mathrm{A}(\mathbf{r})$ is defined by

$$\tilde{\mathrm{A}}(\mathbf{k}) = \int_{\mathbb{R}^2} \mathrm{A}(\mathbf{r}) e^{-j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}. \qquad (1)$$

The wave number spectrum $\tilde{\mathrm{A}}(\mathbf{k})$ may be interpreted as the *global* distribution of the energy of the wave field (as a function of the wave number). Nevertheless, it does not give any information about the *local* distribution of the energy as a function of wave number. On the other hand, the concept of rays is familiar, for instance in geometrical optics. Rays, which should be located at a given point in the space domain, have got a direction or wave number content. Similar descriptions are given in quantum mechanics where both the position and the momentum describe the behaviour of a particle. This justifies the introduction of local spectrum and the use of phase space. This representation implies that the distribution of the energy of the wave field is analysed simultaneously in both space and wave number domains, as a function called local spectrum [1] of four variables (x, y, k_x, k_y). A problem of representing such a function occurs and the choice proposed here is to fix an observation point (x, y) and to represent the result as a function of $(k_x,$

k_y). Local two-dimensional spectra formalism has been set up in signal processing from the early 80's [2, 3].This paper deals with two quadratic distributions : the Husimi distribution H($\mathbf{r}$,$\mathbf{k}$) introduced in quantum mechanical [4] also called spectrogram in signal processing [2] or Mark's physical spectrum in optics [1], and the Pseudo-Wigner-Ville distribution PWVD($\mathbf{r}$,$\mathbf{k}$) [2]. The application of those distributions in planar waveguide propagation allows to show the different waves contributions to the wave field, at a given point of the waveguide. Figure 1b shows the ideal local spectrum for an observation O according to the configuration of figure 1a, where two different rays (A and B) are supposed to meet at point O.

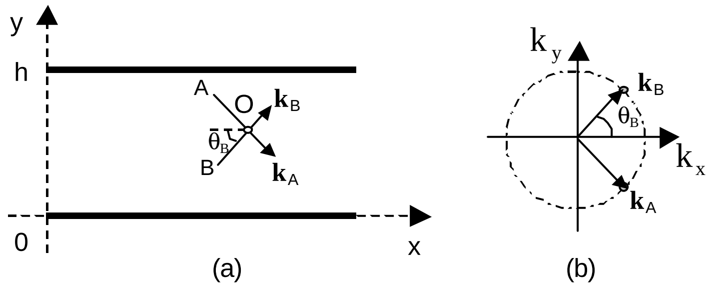

Figure 1 : (a) : Spatial simulation of two rays (A and B) in the planar waveguide which cross at an analysis point O. (b) : Ideal local spectrum showing, in the wave number domain, the contribution of those two rays.

The purpose of the present paper is to suit those local analysis tools to a field created by a line source in a parallel waveguide with perfectly rigid walls. In the high frequency range, the acoustic wave field can be described by geometrical ray description or by modal analysis. Since the cross section of the waveguide is assumed to be large compared to the wavelength, many propagating modes exist, and the field value at a given point of the waveguide is the result of constructive and destructive interferences between the modes. Previous studies of the propagation features of high frequency mode groups have prooved that reinforcement takes place along the ray trajectories [5, 6, 7]. Arbitrarily, we choose in this study to analyse the behaviour of two rays, which represent local plane waves travelling from source to observer, via gaussian modes cluster [8].

II. FORMULATION OF THE PROBLEM

To solve this waveguide problem, we study here the two-dimensional Green's function for a homogeneously filled parallel plane configuration with perfectly rigid walls. We seek a solution of the two-dimensional time harmonic wave equation

$$\left[\Delta + k^2\right] G(\mathbf{r};\mathbf{r_0}) = -d\left(\mathbf{r} - \mathbf{r}_0\right) \qquad (2)$$

satisfied by the field of a line source at $\mathbf{r}_0$, in a parrallel plane waveguide with walls at x = 0, h (Figure 1a) whereon the field satisfies the Neumann boundary conditions. k denotes the free-space wave number and at $|x| \to \infty$, the field satisfies the radiation condition.

II.1. Solution

The Green function can be expressed by the image method or by a modal expansion. The ray field sum can be converted into a sum of guided modes with the help of the Poisson sum formula [7] as illustrated in Figure 2.

II.1.1. Ray Representation

From source to observer an infinite number of multiple reflected rays exists in the waveguide. To describe the wave phenomena, four classes of rays may be distinguished, depending on the directions of departure from the line source and arrival at the observer. The total ray field is written as the sum of the fields of each class as,

$$G(\mathbf{r};\mathbf{r}_0) = \sum_{i=1}^{4}\sum_{n=0}^{+\infty} G_n^{(i)}(\mathbf{r};\mathbf{r}_0) \quad (3.1)$$

$$= \frac{j}{4}\sum_{i=1}^{4}\sum_{n=0}^{+\infty} H_0^{(1)}\left(k\left|\mathbf{r}-\mathbf{r}_n^{(i)}\right|\right), \quad (3.2)$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order 0 and $\mathbf{r}_n^{(i)}$ represents the position of the different sources (real and image). The index (i) and n respectively represent the species of the ray and the number of reflection at upper or lower boundaries [7, 9].

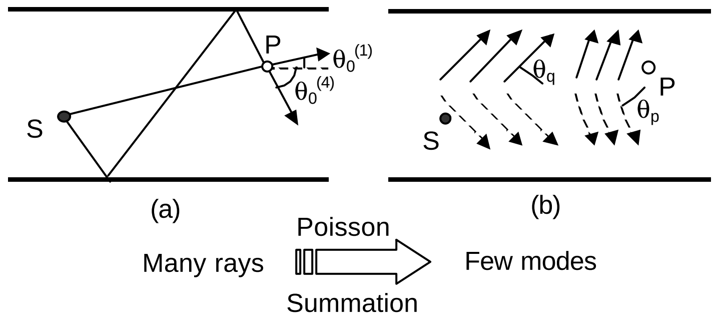

Figure 2 : Many multiple totally reflected rays from the source S to observer P are summed collectively into trapped modes. Two modes (angles θ_q and θ_p) are shown.

II.1.2. Guided Mode Representation

The Green function can be expressed as a sum of guided modes propagating or decaying along x away from the source plane $x = x_0$. The modal expansion is given by [5]:

$$G(\mathbf{r};\mathbf{r}_0) = \sum_{m=0}^{+\infty} \overline{G}_m(\mathbf{r};\mathbf{r}_0) \quad (4.1)$$

$$= \sum_{m=0}^{+\infty} \overline{A}_m e^{jf_m(\mathbf{r};\mathbf{r}_0)}. \quad (4.2)$$

where A_m is the modal excitation coefficient of the mode index m. Because of the Neumann boundary conditions at x = 0 and x = h the mode spectrum is purely discrete with a vertical wave number $k_m = {}^{m\pi}/_h$ with m the mode index and h the distance between the interfaces.

III. SYNTHESIS OF RAY BY A MODAL GAUSSIAN BEAM

Cluster of modes is known to produce an interference maximum along a trajectory (emanating from the source) equivalent to the path of the "modal ray" for the central mode in the group [5, 6]. These concepts, also named spectral filtering [8], are illustrated in the following. Gaussian modal beams are generated in order to have the same path firstly of direct ray $G_0^{(1)}$, and secondly of $G_0^{(4)}$, a ray with one reflection at the upper and lower boundary (Figure 2a).

A mode can thus be regarded as two interfering sets of rays at angles (Figure 2b),

$$\theta_m = \sin^{-1}\left(\frac{m\pi}{kh}\right) \quad (5)$$

and this interpretation has been widely used. In addition the departure angles corresponding to ray species i=1 and i=4 are respectively given by,

$$\theta_n^{(i)} = \tan^{-1}\left(\frac{y-(2nh+y_0)}{x-x_0}\right), \quad (6)$$

where for i = 1 the index $n \in \mathbb{Z}^{-*}$ and for i = 4 the index $n \in \mathbb{Z}^{*}$. The couples (x_0,y_0) and (x,y) are respectively the coordinates of source and observer. According to the KAMEL and FELSEN criterion [6] (the subscript (i) will be omitted entirely henceforth) the ray equivalent of a group of modes is introduced for a mode bundle $M_1 < m < M_2$, which should be chosen so that only one single ray

G_n is contained within it. $M_{1,2}$ is chosen so that the rays in the last modal congruences have angles lying approximately halfway between the angles of G_n and $G_{n\mp1}$, respectively:

$$q_{M_{1,2}} \approx \frac{(q_n + q_{n\pm1})}{2}. \tag{7}$$

III.1. Numerical Example

The source point and the observer have respectively the coordinates $x_0 = 0$, $y_0 = 11.52\ \lambda$ and x = 69.58 λ, y = 35.56 λ with λ the wavelenght of the source. The duct height is h = 51.54 λ and allows 103 propagative modes. The mode which upward congruence is the closest to the $\theta_0^{(1)}$ = 19.11° departure angle yields m = 34, and the mode which downward congruence is the closest to the $\theta_0^{(4)}$ = -48.62° departure angle yields m = 77. We denote the best mode bundle widths for the species i = 1 and i = 4 as follows, $\Delta M^{(1)}$ and $\Delta M^{(2)}$. The distribution of modal eigenangles θ_m is dense near θ ~ 0 and sparse near cutoff (θ ~ π/2), in contrast with $\theta_n^{(1)}$ and $\theta_n^{(4)}$ (see Figure 3 where a cross (×) represents an eigenangle θ_m and a black dot (•) represents a geometrical angle of a ray θ_n.)

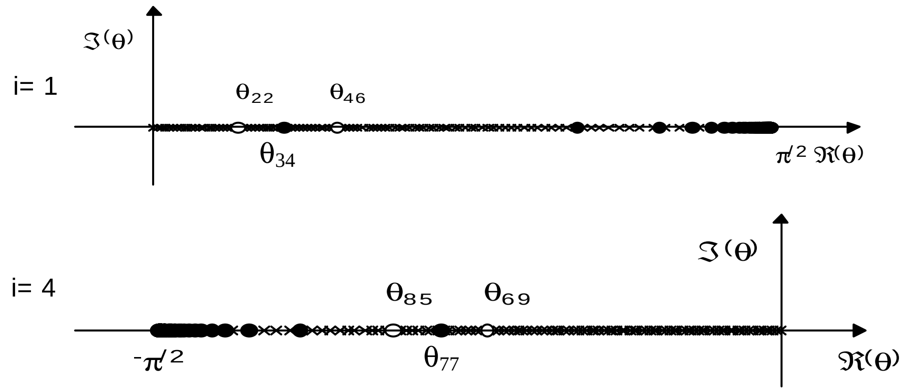

Figure 3: Locations of mode and ray angles θ_m (×) and θ_n (•) for the mode bundle defined in (7). Mode bundles are represented by white dots (o)

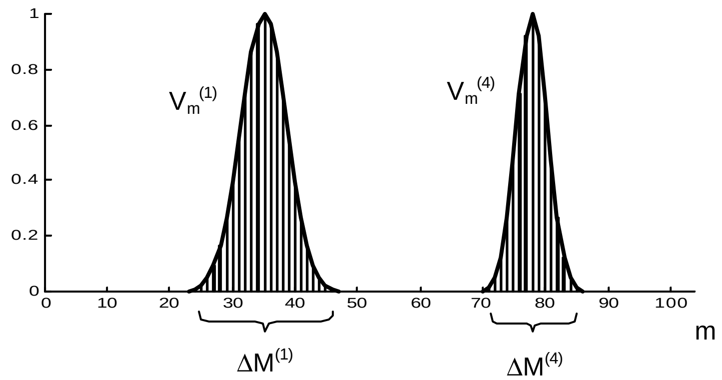

Figure 4: Plot of the gaussian envelope $V_m^{(1)}$ and $V_m^{(4)}$ of the mode bundles $\Delta M^{(1)}$ and $\Delta M^{(4)}$ according to the mode index m.

For species i = 1 and i = 4, the mode bundles are respectively, $\Delta M^{(1)}$ = 25 modes and $\Delta M^{(4)}$= 17 modes. The mode bundles are weighed by a gaussian window [8] in contrast to a rectangular window [5, 6]. The weight of each mode is represented in Figure 4 where their location is represented by a vertical line, which amplitude is indicated by the length of the line. Simulation shown in Figure 5 proves that a modal gaussian beam interferes constructively when a reflected ray can be contained within it. The field of the two modal gaussian beams may be expressed

$$F(\mathbf{r}) = F^{(1)}(\mathbf{r};\mathbf{r}_0) + F^{(4)}(\mathbf{r};\mathbf{r}_0) \tag{8.1}$$

$$= \sum_{m=22}^{46} V_m^{(1)} \overline{G}_m(\mathbf{r};\mathbf{r}_0) + \sum_{m=69}^{85} V_m^{(4)} \overline{G}_m(\mathbf{r};\mathbf{r}_0). \tag{8.2}$$

according to,

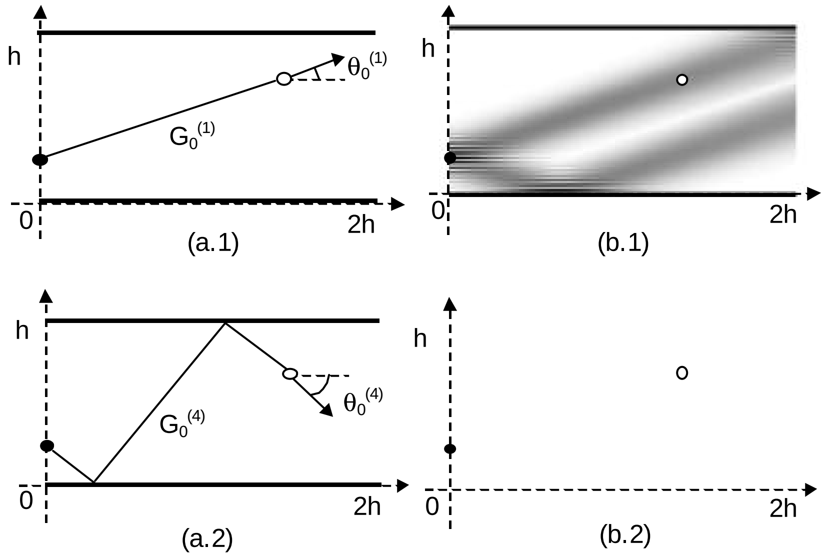

Figure 5: Ray path of $G_0^{(1)}$ (a.1) and $G_0^{(4)}$ (a.2), and modulus of the pressure field introduced by the modal gaussian beam $F^{(1)}(\mathbf{r})$ (b.1) and $F^{(4)\,)}(\mathbf{r})$ (b.2).

IV. PHASE SPACE ANALYSIS

The idea of representing a field with an energy distribution function in the « space–wave number » domain is generaly associated with a localization problem in phase space $(\mathbf{r},\mathbf{k})$. This point of view introduces the incertainty principle between the space domain and the wave number domain. The description of a multicomponent harmonic signal in the conjoint « space–wave number » domain allows especially to interpret the images ((k_x, k_y) spectrum) as local wave vectors [2, 3].

IV.1. The Husimi Distribution

The Husimi distribution (H) is defined as follow,

$$H_F(\mathbf{r},\mathbf{k}) = \left| \int_{\mathbb{R}^2} F(\mathbf{v}) W^*(\mathbf{r}-\mathbf{v}) e^{-j\mathbf{k}\cdot\mathbf{v}} d\mathbf{v} \right|^2, \tag{9}$$

where $*$ denotes complex conjugation, and $W(\mathbf{r})$ is a two dimensional gaussian window. This method is limited by the compromise between spatial and spectral resolution. The improvement of the spectral image can be done only with a widening of the window function.

IV.2. The Pseudo Wigner Ville Distribution

The Pseudo Wigner Ville distribution (PWVD) is defined by,

$$PWVD_F(\mathbf{r},\mathbf{k}) = \int_{\mathbb{R}^2} \left| W^*\left(\frac{\boldsymbol{\alpha}}{2}\right) \right|^2 F\left(\mathbf{r}+\frac{\boldsymbol{\alpha}}{2}\right) F^*\left(\mathbf{r}-\frac{\boldsymbol{\alpha}}{2}\right) e^{-j\mathbf{k}\cdot\boldsymbol{\alpha}} d\boldsymbol{\alpha}. \tag{10}$$

The PWVD involves the use of bounds of integration in contrast to the Wigner distribution (WD) with the help of a symmetrical window $W(\mathbf{r})$. Consequently, the PWVD may be seen as a smoothing of the WD in the wave number domain only.

IV.3. Numerical Example

Consider the two-dimensional signal obtained by the two modal gaussian beams given by the equation (8). For both distribution functions, the gaussian window is applied such as its widths along x and y axes are $W_x = W_y = 6\ \lambda$. According to the choice of a sampling frequency equal to about 5 times the frequency of the studied signal, the numerical analysis window is $W(29\times29)$. The H and the PWVD distributions are computed at the point $(x, y) = (69.58\lambda, 35.56\lambda)$ with an 512 points FFT algorithm, and the results are presented in Figure 6. The dashed circle correspond to the radiation circle in the wave number domain introduced by the point source.

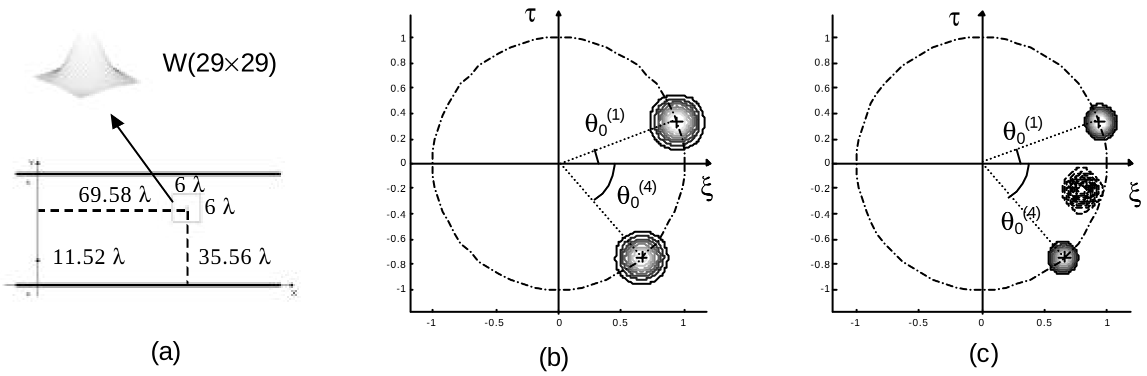

Figure 6: Local normalized wave number representation (ξ, τ) of a two dimensional signal composed with two modal gaussian beam. (a) configuration of the local analysis, (b) Husimi distribution, (c) Pseudo Wigner Ville distribution.

V. CONCLUSION

We have studied here the propagation problem of a point source between two infinite perfect walls. Mode bundles are a good approximation to find propagation channel in the waveguide which have the same path of rays. Two of them are treated more particularly. The introduction of phase space distribution function give a local description of a field, in the wave number domain, at a given point of the waveguide. It is well suited to the analysis of the local plane wave behaviour of two modal gaussian beams. It particularly gives the direction of the trajectories of propagation perpendicular to the local wave front of those two gaussian beams at an analysis point.

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